

Reconstruction of the optical potential from scattering data

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Abstract

We propose a method for reconstruction of the optical potential from scattering data. The algorithm is a two-step procedure. In the first step the real part of the potential is determined analytically via solution of the Marchenko equation. At this point we use a diagonal Padé approximant of the corresponding unitary S -matrix. In the second step the imaginary part of the potential is determined via the phase equation of the variable phase approach. We assume that the real and the imaginary parts of the optical potential are proportional. We use the phase equation to calculate the proportionality coefficient. A numerical algorithm is developed for a single and for coupled partial waves. The developed procedure is applied to analysis of 1S_0 NN , 3SD_1 NN , $P31$ π^-N and $S01$ K^+N data.

1 Introduction

A lot of developments and applications of the classical approaches of Gelfand, Levitan [1] and Marchenko [2] for the solution of the inverse-scattering problem at fixed angular momentum exist, and we have several of excellent reviews on the subject [3, 4]. The direct application of these approaches to construction of local two-body potentials which are phase equivalent to the effective potentials occurring in theories describing reactions of composite particles is impossible. For, such potential must be complex in order to reproduce the loss of flux above the inelastic threshold. But it must reproduce the real phase shifts below this threshold and must be real itself. These requirements are incompatible for potentials being energy independent by construction. For very low threshold these approaches are applicable and produce energy independent complex potentials [5, 6]. In the general case the empirical energy dependent optical potentials are usually inferred by fitting of the parameters of an assumed analytic potential [7, 8]. This approach has two major shortcomings: a complexity and inconvenience of fitting simultaneously many nonlinear parameters; and lack of correlation of the parameters obtained at various energies.

In this paper we develop an inversion method that is free of these shortcomings. The method is based on a fixed- l inverse scattering theory and on a special parameterization of the optical potential. The proposed procedure is a two-step process. In the first step the phase shift data are used to determine a real potential via solution of the Marchenko equation. At this point we use a diagonal Padé approximant $[M/M]$ of the corresponding unitary S -matrix. In the second step the imaginary part of the potential is determined via the phase equation of the variable phase approach [13]. We assume that the real and the imaginary parts of the optical potential are proportional. The value of the proportionality coefficient is predicted by the phase equation and is refined by the iterative algorithm. We develop this method for single and for coupled partial waves. The whole procedure is applied to analyze 1S_0 NN , 3SD_1 NN data and to analyze $P31$ π^-N and $S01$ K^+N data. These analyses demonstrate that prediction for the proportionality coefficient from the phase equation is very close to a precise value that reproduce the experimental loss of flux.

The plan of the paper is as follows. In Sect. 2 we describe the inverse scattering techniques based on the Marchenko integral equation. The used diagonal Padé approximants of the corresponding S -matrix allow an analytical solution of the Marchenko integral equation [2, 4]. For single partial wave the general solution was presented in [5, 9]. We present a solution for coupled partial waves. These techniques produce real local potentials from phase shift analysis data. In Sect. 3 we consider the phase equation. We investigate how the S -matrix is changing with certain change of the potential. This consideration shows advantages of proportionality of the real and imaginary parts of the optical potential. In Sect. 4 the feasibility of the method is shown in the examples of analyses of NN , π^-N and K^+N scattering data.

2 Inversion algorithm

The Marchenko inverse scattering theory is viewed in detail in Refs. [2, 3, 4]. We shall, therefore, only briefly describe this formalism.

The input data of the Marchenko inversion are

$$\{S(q), (0 < q < +\infty), \tilde{q}_j, M_j, j = 1, \dots, n_b\}, \quad (1)$$

where $S(q)$ is the scattering matrix dependant on the relative momentum q , $q^2 = Em$, $\tilde{q}_j^2 = mE_j \leq 0$, E_j is the energy of the j -th bound state, so that $i\tilde{q}_j \geq 0$, m is the reduced mass. The M_j matrices give the asymptotic behavior of the corresponding normalized bound states.

We proceed from the Marchenko equation for single channel

$$F(x, y) + L(x, y) + \int_x^{+\infty} L(x, t)F(t, y)dt = 0, \quad (2)$$

where the input kernel is given by

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h_l^+(qx) (I - S(q)) h_l^+(qy) dq + \sum_{j=1}^{n_b} M_j^2 h_l^+(iq_j x) h_j^+(iq_j y), \quad (3)$$

$h_l^+(z)$ are the Riccati-Hankel functions.

The output kernel $L(x, y)$ gives the reconstructed potential

$$V(r) = -\frac{dL(r, r)}{dr}. \quad (4)$$

This local energy independent operator $V(r)$ links the Marchenko equation (2) and the radial Schrödinger equation of a fixed angular momentum,

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r) \right] \psi(r, q) = q^2 \psi(r, q). \quad (5)$$

The scattering matrix $S(q)$, matrices M_j and energies E_j are the output data of the direct scattering problem associated with the Schrödinger equation (5).

It has been known for several decades now that S matrices rational in q (ratio of polynomials) correspond to potentials known as Bargmann potentials expressible in terms of the elementary functions [2, 3, 4]. Such fraction may have the same truncated Taylor series as the S matrix it represents. It is then called a Padé approximant. Conjectures and theorems concerning the convergence and analytic continuation properties of Padé approximants are collected in [10]. For single partial wave the general solution of the Marchenko equation via Padé approximant of the S -matrix was presented in [5] and in [9]. We shall, therefore, only present it briefly and turn to the case of coupled partial waves.

A diagonal Padé approximant of the S -matrix is given by

$$S(q) = e^{2i\delta} = \frac{f_2(q) - i f_1(q)}{f_2(q) + i f_1(q)} \quad (6)$$

$f_1(q)$ and $f_2(q)$ are an odd and even polynomials of q , which do not turn to zero at the real axis simultaneously.

This approximant leads to the following expression for the phase shifts $\delta(q)$

$$tg(-\delta(q)) = \frac{f_1(q)}{f_2(q)} \quad (7)$$

Inasmuch as $\lim_{q \rightarrow \infty} \delta(q) \sim 1/q$ for regular potentials, it is evident that degree of $f_1(q)$ must be less than degree of $f_2(q)$ by 1. Let us select N discrete momenta q_i such that the corresponding $\delta(q_i) = \delta_i$ are known. Use of these values in eq. (7) transforms the latter into a set of inhomogeneous linear equations from which N coefficients of the polynomials $f_1(q)$ and $f_2(q)$ can be determined. This is a usual strategy

[5], but since any set of q_i is experimentally limited from above ($q_i < q_{max}$) there is some uncertainty in determination of $S(q)$. Even if a set of q_i is dense and the agreement between the data and the used approximant is excellent the arbitrary behavior of $S(q)$ above q_{max} guarantees that a solution of the inverse problem is arbitrary as well. We assume that about and above q_{max} the true $\delta(q)$ depends only slightly on details of the potential and does not depend on its asymptotic at $r > r_{max}$. Then we take a model potential (in our calculations $V_{model}(r) = A \exp(-br)$) and fit parameters (A and b) so that

$$\delta_{model}(\approx q_{max}) \approx \delta(\approx q_{max}), \quad (8)$$

here signs \approx mean that about $q = q_{max}$ the chosen model potential gives a phase curve which goes inside error bars. This means that we take $\delta_{model}(q)$ as an asymptotic for $\delta(q)$ when $q > q_{max}$. In the segment $[0, q_{max}]$ the $\delta(q)$ is defined by some spline that approximates the data points $\{q_i, \delta_i\}$. In this way we may control the Padé fit in the line segment $[0, Q_{max}]$, where Q_{max} is arbitrary and is fixed by convergence of the whole inversion procedure. The needed accuracy of approximant is attained by increasing of N which in turn defines degrees of $f_1(q)$ and $f_2(q)$.

Approximant (7) leads to a degenerate input kernel $F(x, y)$. We calculate the integral in eq. (3) using the residue theorem. For approximant (6) the result of the integration is

$$\begin{aligned} F(x, y) &= i \sum_{i=1}^{n_{pos}} Res [h_l^+(qx) (I - S(q)) h_l^+(qy)] \Big|_{q=\beta_i} + \sum_{i=1}^{n_b} M_i^2 h_l^+(\tilde{q}_i x) h_l^+(\tilde{q}_i y) = \\ &= \sum_{i=1}^{n_{pos}} b_i h_l^+(\beta_i x) h_l^+(\beta_i y) + \sum_{i=1}^{n_b} M_i^2 h_l^+(\tilde{q}_i x) h_l^+(\tilde{q}_i y) = \sum_{j=1}^n b_j h_l^+(\beta_j x) h_l^+(\beta_j y), \end{aligned} \quad (9)$$

where β_i ($i = 1, \dots, n_{pos}$) are all S -matrix poles with $\Im \beta_i > 0$, $\beta = \{\beta_1, \dots, \beta_{n_{pos}}, \tilde{q}_1, \dots, \tilde{q}_{n_b}\}$, $n = n_{pos} + n_b$. We assume that all poles are of first order so that

$$\begin{aligned} Res [h_l^+(qx) (I - S(q)) h_l^+(qy)] \Big|_{q=\beta_i} &= 2i Res \left[h_l^+(qx) \frac{f_1(q)}{f_2(q) + i f_1(q)} h_l^+(qy) \right] \Big|_{q=\beta_i} = \\ &= 2i \frac{f_1(\beta_i)}{f_2'(\beta_i) + i f_1'(\beta_i)} h_l^+(\beta_i x) h_l^+(\beta_i y) = b_i h_l^+(\beta_i x) h_l^+(\beta_i y), \end{aligned} \quad (10)$$

here we have denoted $f_i'(q) = df_i(q)/dq$, ($i = 1, 2$).

In this case the input kernel of eq. (2) is a degenerate one as well as its output kernel

$$L(x, y) = \sum_{i=1}^n P_i(x) h_l^+(\beta_i y), \quad (11)$$

where $P_i(x)$ are unknown coefficients. Substitution of (9) and (11) into (2) yields

$$\sum_{i=1}^n h_l^+(\beta_i y) \left(b_i h_l^+(\beta_i x) + P_i(x) + b_i \sum_{k=1}^n P_k(x) \int_x^{+\infty} h_l^+(\beta_k t) h_l^+(\beta_i t) dt \right) = 0. \quad (12)$$

Linear independence of the $h_l^+(\beta_i y)$ implies that

$$b_i h_l^+(\beta_i x) + P_i(x) + b_i \sum_{k=1}^n P_k(x) \int_x^{+\infty} h_l^+(\beta_k t) h_l^+(\beta_i t) dt = 0, \quad (13)$$

or

$$\sum_{k=1}^n A_{ik}(x) P_k(x) = D_i(x) \quad (i = 1, \dots, n), \quad (14)$$

where $D_i(x) = -b_i h_l^+(\beta_i x)$ and after applying Riccati-Hankel integration formulas (see Appendix) in (13) we have

$$A_{ik} = \begin{cases} 1 + b_i x ((h_l^+(\beta_i x))^2 - h_{l-1}^+(\beta_i x) h_{l+1}^+(\beta_i x))/2, & \text{for } i = k \\ -b_i \frac{\beta_i h_{l-1}^+(\beta_i x) h_l^+(\beta_k x) - \beta_k h_l^+(\beta_i x) h_{l-1}^+(\beta_k x)}{\beta_i^2 - \beta_k^2}, & \text{for } i \neq k. \end{cases} \quad (15)$$

The functional coefficients $P_i(x)$ are defined by (14)

$$P_i(x) = (A^{-1}D)_i. \quad (16)$$

Finally we derive $L(x, y)$ and the potential $V(r)$ from (11) and (4).

In case of two coupled channels we present only sketchy derivations because of their awkwardness. In this case the system of the partial Schrödinger equations is

$$\left(\frac{d^2}{dr^2} + V(r) + \begin{pmatrix} \frac{l_1(l_1+1)}{r^2} & 0 \\ 0 & \frac{l_2(l_2+1)}{r^2} \end{pmatrix} \right) \begin{pmatrix} \chi_1(r) \\ \chi_2(r) \end{pmatrix} = q^2 \begin{pmatrix} \chi_1(r) \\ \chi_2(r) \end{pmatrix}, \quad (17)$$

$$V(r) = \begin{pmatrix} V_1(r) & V_T(r) \\ V_T(r) & V_2(r) \end{pmatrix}, \quad (18)$$

where $V_1(r)$, $V_2(r)$ are potentials in channels 1 and 2, $V_T(r)$ is potential coupling them, $\chi_1(r)$ and $\chi_2(r)$ are channel wave functions.

By analogy with (6) we approximate the S -matrix by the following expression

$$S(x) = \begin{pmatrix} \exp(2i\delta_1) \cos 2\varepsilon & i \exp(i(\delta_1 + \delta_2)) \sin 2\varepsilon \\ i \exp(i(\delta_1 + \delta_2)) \sin 2\varepsilon & \exp(2i\delta_2) \cos 2\varepsilon \end{pmatrix} =$$

$$= \begin{pmatrix} \left(\frac{f_2^{(1)}(q) - i f_1^{(1)}(q)}{f_2^{(1)}(q) + i f_1^{(1)}(q)} \right)^2 \frac{(f_2^{(12)}(q))^2 - (f_1^{(12)}(q))^2}{(f_2^{(12)}(q))^2 + (f_1^{(12)}(q))^2} & -2i \frac{f_2^{(12)}(q) f_1^{(12)}(q)}{(f_2^{(12)}(q))^2 + (f_1^{(12)}(q))^2} \prod_{j=1,2} \frac{f_2^{(j)}(q) - i f_1^{(j)}(q)}{f_2^{(j)}(q) + i f_1^{(j)}(q)} \\ -2i \frac{f_2^{(12)}(q) f_1^{(12)}(q)}{(f_2^{(12)}(q))^2 + (f_1^{(12)}(q))^2} \prod_{j=1,2} \frac{f_2^{(j)}(q) - i f_1^{(j)}(q)}{f_2^{(j)}(q) + i f_1^{(j)}(q)} & \left(\frac{f_2^{(2)}(q) - i f_1^{(2)}(q)}{f_2^{(2)}(q) + i f_1^{(2)}(q)} \right)^2 \frac{(f_2^{(12)}(q))^2 - (f_1^{(12)}(q))^2}{(f_2^{(12)}(q))^2 + (f_1^{(12)}(q))^2} \end{pmatrix} \quad (19)$$

This is again the most general Padé approximant for the S -matrix. It was used in [11] in an other form, but corresponding analytical solution of the inverse scattering problem was not presented.

The coefficients of this Padé approximant are determined from the equations analogous to (20)

$$tg \left(-\frac{\delta_i(q)}{2} \right) = \frac{f_1^{(i)}(q)}{f_2^{(i)}(q)}, \quad i = 1, 2 \quad (20)$$

and

$$tg(\varepsilon(q)) = \frac{f_1(q)}{f_2(q)}. \quad (21)$$

The generalized Marchenko equation for coupled channels formally has the former view [12]

$$L(x, y) + F(x, y) + \int_x^{+\infty} L(x, t) F(t, y) dt = 0, \quad (22)$$

but functions involved are matrices (2×2)

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(qx) [I - S(q)] H(qy) dq + \sum_{i=1}^{n_b} H(\beta_i x) M_i H(\beta_i y), \quad (23)$$

where

$$H(x) = \begin{pmatrix} h_{l_1}^+(x) & 0 \\ 0 & h_{l_2}^+(x) \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (24)$$

Insertion of (19) into (23) and applying of the residue theorem yields

$$F(x, y) = i \sum_{i=1}^{n_{pos}} Res [H(qx) (I - S(q)) H(qy)]|_{q=\beta_i} + \sum_{i=1}^{n_b} H(\beta_i x) M_i^2 H(\beta_i y) =$$

$$= \sum_{i=1}^n H(\beta_i x) Q_i^1 H(\beta_i y) + \sum_{i=1}^{n_{pos}^{(2)}} x H'(\beta_i x) Q_i^2 H(\beta_i y) + \sum_{i=1}^{n_{pos}^{(2)}} H(\beta_i x) Q_i^2 H'(\beta_i y) y, \quad (25)$$

where β_i ($i = 1, \dots, n_{\text{pos}}$) are all S -matrix poles with $\Im \beta_i > 0$, β_i ($i = 1, \dots, n_{\text{pos}}^{(2)}$) are poles of the second order, $\beta = \{\beta_1, \dots, \beta_{n_{\text{pos}}}, \tilde{q}_1, \dots, \tilde{q}_{n_b}\}$, $n = n_{\text{pos}} + n_b$,

$$H'(x) = \begin{pmatrix} dh_{l_1}^+(x)/dx & 0 \\ 0 & dh_{l_2}^+(x)/dx \end{pmatrix}.$$

We note that there are poles of the first as well as of the second order in the diagonal matrix elements and there are poles only of the first order in the off-diagonal matrix elements. Poles of the second order in the diagonal elements are poles of the first order in the off-diagonal matrix elements, and they must be enumerated twice. Q_i^j ($j = 1, 2$) are constant matrices they are trivial but cumbersome therefore, we do not give them.

We solve eq. (22) using substitution

$$L(x, y) = \sum_{i=1}^n P_i(x) H(\beta_i y) + \sum_{i=1}^n N_i(x) y H'(\beta_i y), \quad (26)$$

where $P_i(x)$, $N_i(x)$ are unknown functional (2×2) matrix-coefficients. Linear independence of the $H(\beta_i y)$ and $y H'(\beta_i y)$ implies that

$$\begin{aligned} \sum_i P_i(x) Q_{ij}^3(x) + \sum_i N_i(x) Q_{ij}^5(x) &= H(\beta_j x) Q_j^1 + x H'(\beta_j x) Q_j^2 \\ \sum_i N_i(x) Q_{ij}^6(x) + \sum_i P_i(x) Q_{ij}^4(x) &= H(\beta_j x) Q_j^2 \end{aligned} \quad (27)$$

where

$$\begin{aligned} Q_{ij}^3(x) &= I\delta_{ij} + \int_x^{+\infty} H(\beta_i t) H(\beta_j t) dt \times Q_j^1 + \int_x^{+\infty} t H(\beta_i t) H'(\beta_j t) dt \times Q_j^2 \\ Q_{ij}^4(x) &= \int_x^{+\infty} H(\beta_i t) H(\beta_j t) dt \times Q_j^2 \\ Q_{ij}^5(x) &= \int_x^{+\infty} t H'(\beta_i t) H(\beta_j t) dt \times Q_j^1 + \int_x^{+\infty} t^2 H'(\beta_i t) H'(\beta_j t) dt \times Q_j^2 \\ Q_{ij}^6(x) &= I\delta_{ij} + \int_x^{+\infty} t H'(\beta_i t) H(\beta_j t) dt \times Q_j^2, \end{aligned} \quad (28)$$

Integrals of expressions (28) are presented in Appendix. Matrix equations (27) can be trivially reduced to scalar linear equations. Having solved this linear equation system we get the sought-for potential from (26) and (4).

The multichannel generalization is trivial.

3 The optical potential

In this section we consider changes of S -matrix that are induced by certain transformation of real potential.

First we consider the one channel problem.

The phase equation [13] for the initial potential $V^0(r)$ obtained by some inversion procedure (from Marchenko equation in our calculations) is

$$\delta_l^{(0)} = -\frac{1}{q} \int_0^\infty V^{(0)}(r) \hat{D}_l^2(qr) \sin^2(\delta_l(qr) + \delta^{(0)}(r)) dr, \quad (29)$$

where $\hat{D}_l(z)$ and $\hat{\delta}_l(z)$ are Riccati-Bessel amplitude and phase correspondingly [13]

$$\hat{D}_l(x) = \sqrt{j_l^2(x) + n_l^2(x)}, \quad (30)$$

$$\hat{\delta}_l(x) = -\arctan(j_l(x)/n_l(x)) \quad (31)$$

Let us consider the complex-valued potential $V^{(1)}(r)$ obtained from $V^0(r)$ by transformation

$$V^{(1)}(r) = (1 + i\alpha) V^{(0)}(r), \quad (32)$$

where α is some real parameter. Such parametrization was used in [14] but without analysis (α was fitted). Evidently the phase equation for this potential is

$$\delta^{(1)} = -\frac{1}{q} (1 + i\alpha) \int_0^\infty V^{(0)}(r) \hat{D}_l^2(qr) \sin^2(\hat{\delta}_l(qr) + \delta^{(1)}(r)) dr. \quad (33)$$

From eqs. (29) and (33) we get

$$\begin{aligned} \delta^{(1)} - (1 + i\alpha) \delta^{(0)} &= \\ &= -\frac{1+i\alpha}{q} \int_0^\infty V^{(0)}(r) \hat{D}_l^2(qr) \left(\sin^2(\hat{\delta}_l(qr) + \delta^{(1)}(r)) - \sin^2(\hat{\delta}_l(qr) + \delta^{(0)}(r)) \right) dr = \\ &= -\frac{1+i\alpha}{q} \int_0^\infty V^{(0)}(r) \hat{D}_l^2(qr) \sin(2\hat{\delta}_l(qr) + \delta^{(1)}(r) + \delta^{(0)}(r)) \sin(\delta^{(1)}(r) - \delta^{(0)}(r)) dr \end{aligned} \quad (34)$$

For smooth enough potentials the right side of eq. (34) rapidly decreases comparing with $\delta^{(0)}$ and $\delta^{(1)}$, because there is a rapidly oscillating around zero function under the integral in (34) (underlined). Its frequency behaves as $2q$ for big q (see (31)). Then as the first approximation we may take

$$\delta^{(1)} \approx (1 + i\alpha) \delta^{(0)} = \delta_R + i\delta_I. \quad (35)$$

For inelastic scattering the S -matrix is expressed through the real inelasticity parameter ρ and the real phase shift δ

$$S = \cos^2(\rho) e^{2i\delta} = e^{2i(\delta_R + i\delta_I)}, \quad (36)$$

so we easily arrive at

$$\delta_R = \delta \approx \delta^{(0)}, \quad (37)$$

$$\cos^2(\rho) \approx e^{-2\alpha\delta^{(0)}}, \quad (38)$$

whence it follows that $\alpha\delta \geq 0$. The formula (38) allows to calculate the parameter α from the known values ρ and $\delta^{(0)} \approx \delta$.

Eqs. (34-35) imply that

$$\int_0^\infty V^{(0)}(r) \hat{D}_l^2(qr) \sin^2(\hat{\delta}_l(qr) + \delta^{(0)}(r)) dr \simeq \int_0^\infty V^{(0)}(r) \hat{D}_l^2(qr) \sin^2(\hat{\delta}_l(qr) + \delta^{(1)}(r)) dr, \quad (39)$$

Consideration of the coupled partial waves is more complicated. The initial real potential is

$$V^{(0)}(r) = \begin{pmatrix} V_1^{(0)} & V_T^{(0)} \\ V_T^{(0)} & V_2^{(0)} \end{pmatrix}. \quad (40)$$

The equations for eigenphases and mixing parameter of potential (40) are [13]

$$\delta_1^{(1)} = I_{11}^{(0)} + I_{12}^{(0)} + I_{13}^{(0)} \quad (41)$$

$$\delta_2^{(1)} = I_{21}^{(0)} + I_{22}^{(0)} + I_{23}^{(0)} \quad (42)$$

$$\epsilon^{(1)} = I_{31}^{(0)} + I_{32}^{(0)} + I_{33}^{(0)}, \quad (43)$$

where

$$\begin{aligned}
I_{11}^{(0)} &= -\frac{1}{q} \int_0^\infty dr V_1^{(0)}(r) \cos^2 \epsilon^{(0)}(r) \hat{D}_{l_1}^2(qr) \sin^2(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)) \\
I_{12}^{(0)} &= -\frac{1}{q} \int_0^\infty dr V_2^{(0)}(r) \sin^2 \epsilon^{(0)}(r) \hat{D}_{l_2}^2(qr) \sin^2(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)) \\
I_{13}^{(0)} &= -\frac{1}{q} \int_0^\infty dr V_T^{(0)}(r) \sin 2\epsilon^{(0)}(r) \hat{D}_{l_2}(qr) \sin(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)) \hat{D}_{l_1}(qr) \sin(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)) \quad (44)
\end{aligned}$$

$$\begin{aligned}
I_{21}^{(0)} &= -\frac{1}{q} \int_0^\infty dr V_1^{(0)}(r) \sin^2 \epsilon^{(0)}(r) \hat{D}_{l_1}^2(qr) \sin^2(\hat{\delta}_{l_1}(qr) + \delta_2^{(0)}(r)) \\
I_{22}^{(0)} &= -\frac{1}{q} \int_0^\infty dr V_2^{(0)}(r) \cos^2 \epsilon^{(0)}(r) \hat{D}_{l_2}^2(qr) \sin^2(\hat{\delta}_{l_2}(qr) + \delta_2^{(0)}(r)) \\
I_{23}^{(0)} &= -\frac{1}{q} \int_0^\infty dr V_T^{(0)}(r) \sin 2\epsilon^{(0)}(r) \hat{D}_{l_2}(qr) \sin(\hat{\delta}_{l_2}(qr) + \delta_2^{(0)}(r)) \hat{D}_{l_1}(qr) \sin(\hat{\delta}_{l_1}(qr) + \delta_2^{(0)}(r)) \quad (45)
\end{aligned}$$

$$\begin{aligned}
I_{31}^{(0)} &= \frac{1}{2q} \int_0^\infty \frac{\sin 2\epsilon^{(0)}(r) dr}{\sin(\delta_1^{(0)}(r) - \delta_2^{(0)}(r))} V_1^{(0)}(r) \hat{D}_{l_1}^2(qr) \sin(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)) \sin(\hat{\delta}_{l_1}(qr) + \delta_2^{(0)}(r)) \\
I_{31}^{(0)} &= -\frac{1}{2q} \int_0^\infty \frac{\sin 2\epsilon^{(0)}(r) dr}{\sin(\delta_1^{(0)}(r) - \delta_2^{(0)}(r))} V_2^{(0)}(r) \hat{D}_{l_2}^2(qr) \sin(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)) \sin(\hat{\delta}_{l_2}(qr) + \delta_1^{(0)}(r)) \\
I_{31}^{(0)} &= -\frac{1}{2q} \int_0^\infty \frac{V_T^{(0)}(r) \hat{D}_{l_1}(qr) \hat{D}_{l_2}(qr) dr}{\sin(\delta_1^{(0)}(r) - \delta_2^{(0)}(r))} \left[\cos 2\epsilon^{(0)}(r) \sin(\hat{\delta}_{l_1}(qr) + \delta_1^{(0)}(r)) \sin(\hat{\delta}_{l_2}(qr) + \delta_2^{(0)}(r)) - \right. \\
&\quad \left. -\frac{1}{2} \left(\cos 2\epsilon^{(0)}(r) - 1 \right) \sin(\delta_1^{(0)}(r) - \delta_2^{(0)}(r)) \sin \left(\hat{\delta}_{l_1}(qr) - \hat{\delta}_{l_2}(qr) \right) \right] \quad (46)
\end{aligned}$$

By analogy with the one channel case the following generalization for the optical potential is derived

$$V^{(1)}(r) = \begin{pmatrix} (1 + i\alpha_1) V_1^{(0)} & (1 + i\alpha_3) V_T^{(0)} \\ (1 + i\alpha_3) V_T^{(0)} & (1 + i\alpha_2) V_2^{(0)} \end{pmatrix}. \quad (47)$$

Evidently the phase equations for this potential is

$$\delta_1^{(1)} = (1 + i\alpha_1) I_{11}^{(1)} + (1 + i\alpha_2) I_{12}^{(1)} + (1 + i\alpha_3) I_{13}^{(1)} \quad (48)$$

$$\delta_2^{(1)} = (1 + i\alpha_1) I_{21}^{(1)} + (1 + i\alpha_2) I_{22}^{(1)} + (1 + i\alpha_3) I_{23}^{(1)} \quad (49)$$

$$\epsilon^{(1)} = (1 + i\alpha_1) I_{31}^{(1)} + (1 + i\alpha_2) I_{32}^{(1)} + (1 + i\alpha_3) I_{33}^{(1)} \quad (50)$$

Integrals $I_{ij}^{(1)}$ are defined as $I_{ij}^{(0)}$ in (44-46) but through $\delta_1^{(1)}(r)$, $\delta_2^{(1)}(r)$ and $\epsilon^{(1)}(r)$ instead of $\delta_1^{(0)}(r)$, $\delta_2^{(0)}(r)$ and $\epsilon^{(0)}(r)$.

Evidently we cannot consider (48-50) in a manner like (34). But we assume that

$$I_{ij}^{(1)} = I_{ij}^{(0)} + \sum_{i,j=1,2,3} o(I_{ij}^{(0)}), \quad (51)$$

where

$$\sum_{i,j=1,2,3} o(I_{ij}^{(0)}) \ll I_{ij}^{(0)}, \quad \text{for } i, j = 1, 2, 3. \quad (52)$$

This assumption can be considered as a generalization of (39). It is hard to prove in the general case, but our calculations show that this is true at least in case of 3SD_1 NN scattering.

Eigenphases $\delta_i^{(0)}$, $i = 1, 2$ and mixing parameter $\hat{\epsilon}^{(0)}$ are real and they define a unitary $S^{(0)}$ -matrix

$$S^{(0)} = \begin{pmatrix} \cos^2 \hat{\epsilon}^{(0)} e^{2i\delta_1^{(0)}} + \sin^2 \hat{\epsilon}^{(0)} e^{2i\delta_2^{(0)}} & \cos \hat{\epsilon}^{(0)} \sin \hat{\epsilon}^{(0)} (e^{2i\delta_1^{(0)}} - e^{2i\delta_2^{(0)}}) \\ \cos \hat{\epsilon}^{(0)} \sin \hat{\epsilon}^{(0)} (e^{2i\delta_1^{(0)}} - e^{2i\delta_2^{(0)}}) & \sin^2 \hat{\epsilon}^{(0)} e^{2i\delta_1^{(0)}} + \cos^2 \hat{\epsilon}^{(0)} e^{2i\delta_2^{(0)}} \end{pmatrix} \quad (53)$$

Eigenphases $\delta_i^{(1)}$ and mixing parameter $\hat{\epsilon}^{(1)}$ are complex but they define $S^{(1)}$ -matrix in the regular way

$$S^{(1)} = \begin{pmatrix} \cos^2 \hat{\epsilon}^{(1)} e^{2i\delta_1^{(1)}} + \sin^2 \hat{\epsilon}^{(1)} e^{2i\delta_2^{(1)}} & \cos \hat{\epsilon}^{(1)} \sin \hat{\epsilon}^{(1)} (e^{2i\delta_1^{(1)}} - e^{2i\delta_2^{(1)}}) \\ \cos \hat{\epsilon}^{(1)} \sin \hat{\epsilon}^{(1)} (e^{2i\delta_1^{(1)}} - e^{2i\delta_2^{(1)}}) & \sin^2 \hat{\epsilon}^{(1)} e^{2i\delta_1^{(1)}} + \cos^2 \hat{\epsilon}^{(1)} e^{2i\delta_2^{(1)}} \end{pmatrix}. \quad (54)$$

The complex eigenphases $\delta_i^{(1)}$ and mixing parameter $\hat{\epsilon}^{(1)}$ are defined by the experimental S -matrix ($S \equiv S^{(1)}$). A direct consequence of (48-51) is

$$\Re \delta_i^{(1)} = \delta_i^{(0)}, \quad i = 1, 2; \quad \Re \hat{\epsilon}^{(1)} = \epsilon^{(0)}. \quad (55)$$

$$\Im \delta_i^{(1)} = \sum_{j=1}^3 \alpha_j I_{ij}^{(0)}, \quad i = 1, 2; \quad \Im \hat{\epsilon}^{(1)} = \sum_{j=1}^3 \alpha_j I_{3j}^{(0)}. \quad (56)$$

We may calculate coefficients $I_{ij}^{(1)} \simeq I_{ij}^{(0)}$ ($i, j = 1, 2, 3$) using (44-46). A simpler method is to use the following implication of (48-51)

$$I_{ij}^{(1)} = \Im \frac{\partial \delta_i^{(1)}}{\partial \alpha_j}, \quad i = 1, 2; \quad I_{3j}^{(1)} = \Im \frac{\partial \hat{\epsilon}^{(1)}}{\partial \alpha_j}. \quad (57)$$

Calculated coefficients may be checked by (41-43). Next, we calculate α_i ($i = 1, 2, 3$) from (48-50).

4 The optical potentials

We apply the developed method of inversion to analysis NN , π^-N and K^+N data up to energies where relativistic effects are essential. We take into account these effects in the frames of relativistic quantum mechanics of systems with a fixed number of particles. The review of this approach can be found in [15]. Here we give only some extracts of it.

The relativistic quantum mechanics of systems consisting of a fixed number of particles is based on the conjecture that the number of particles is constant at not very high energies and on the assumption that the group of invariance for the system under consideration is the Poincare group rather than the Galilei one. A system of two particles is described by the wave function, which is an eigenfunction of the mass operator. In this case we may represent this wave function as a product of the external and internal wave functions [16, 17]. The internal wave function χ is also an eigenfunction of the mass operator and satisfies the following equation

$$\left[\sqrt{\hat{q}^2 + m_1^2} + \sqrt{\hat{q}^2 + m_2^2} + V_{int} \right] \chi = M\chi, \quad (58)$$

where V_{int} is some interaction operator acting only through internal variables (spins and relative momentum), \hat{q} is a momentum operator of one of the particles in the center of masses frame. Rearrangement of (58) gives

$$[\hat{q}^2 + 2mV] \chi = q^2 \chi, \quad (59)$$

where

$$q^2 = \frac{M^2}{4} - \frac{m_1^2 + m_2^2}{2} + \frac{(m_1^2 - m_2^2)^2}{4M^2}, \quad (60)$$

m is taken as a nonrelativistic reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}, \quad (61)$$

V is an operator acting like V_{int} only through internal variables.

In case of two particles with equal masses $m_1 = m_2 \equiv 2m$

$$q^2 = \frac{M^2}{4} - 4m^2. \quad (62)$$

Eq. (59) is identical in form to the Schrödinger equation. The quasicordinate representation corresponds to the realization $\mathbf{q} = -i\frac{\partial}{\partial \mathbf{r}}$, $V = V(\mathbf{r})$. In [18] we showed that this formalism can be easily generalized for the case of inelastic channels, particularly it allows to take into account isobar channels in NN scattering. This formal coincidence allows us to apply our inversion algorithm.

We applied the described algorithm of inversion to reconstruction of the nucleon-nucleon potential. As input data for this reconstruction we used modern phase shift analysis data up to 1100 MeV for 3SD_1 state and up to 3 GeV for 1S_0 state of nucleon-nucleon system [19, 20]. The deuteron properties were taken from [21]. These data allow to construct nucleon-nucleon partial potentials sustaining forbidden bound states (Moscow potential introduced in [22]). Parameters of forbidden bound states for the partial Moscow potentials were chosen to be equal to those of model potentials (see Sect. 2). In this way we constructed the NN optical potentials for 1S_0 and 3SD_1 partial waves. These potentials describe the deuteron properties and the phase shift analysis data. The 1S_0 phase shifts of Moscow potential begin from π . 3S_0 phase shifts of Moscow potential begin from 2π . The mixing parameter ϵ_1 of Moscow potential differs from that of traditional repulsive core potential by sign. The real parts of the constructed partial potentials are presented in fig. 1. The calculated values of deuteron properties are compared with the experimental data [21] in Table 1. Only three parameters are fixed as input data of inversion problem. These parameters are energy, A_S and $\eta_{d/s}$. Figs. 1 and 3 demonstrate how changes of δ influence the partial potential for 1S_0 wave.

As another example of application we analyzed the modern $P31 \pi^- N$ data up to 2 GeV and $S01 K^+ N$ data up to 1 GeV [20] and constructed the corresponding optical potentials. The real parts of the constructed partial potentials for $P31 \pi^- N$ data up to 2 GeV and $S01 K^+ N$ are presented in fig. 2.

From eqs. (38) and (56, 57) we calculated parameters α and α_i , ($i = 1, 2, 3$) which define the imaginary parts of potentials. Our predictions were justified. Calculations with optical and real potentials (fig. 3,7,8) show the validity of (37) and (55). The α 's predicted by (38) may be improved by a simple numerical method. Predicted and improved values of α 's are shown in fig. 4,5,6. In all figures "Calc. I" means calculations from predicted values of (37), "Calc. II" means calculations from refined values. Fig. 5 shows that calculation of α_i , ($i = 1, 2, 3$) from (56 and 57) does not require refinement because it uses more precise values of $I_{ij}^{(1)}$ than those implied by assumption that $I_{ij}^{(1)} = I_{ij}^{(0)}$.

In eq. (36) we use parametrization of the S -matrix from [23] whereas parametrization of partial wave analysis [19] is based on type- K scheme [24, 25]. For uncoupled waves, the S -matrix is given by

$$S = \frac{1 - K_i + iK_r}{1 + K_i - iK_r}, \quad (63)$$

$$\text{where } K_r = \tan \tilde{\delta}, \quad K_i = \tan^2 \tilde{\rho}.$$

So we had to recalculate data of [19] into S -matrix, then into parameters of [23] to get input data of inverse problem. Our results are presented in parametrization of [19].

All potentials and inelasticity multipliers (α 's) may be downloaded from cite www.physics.khstu.ru in numerical form.

5 Conclusions

Let us summarize the results presented in this work. In the first place we mention a presented analytical solution of the Marchenko equation for coupled partial waves in case of diagonal Padé approximant of the

corresponding S -matrix. The inverse scattering scheme at fixed angular momentum is used to construct a local real energy independent potential as a first step of our inversion procedure for single and coupled waves. Furthermore, we consider what changes of S -matrix are induced by certain transformation of real potential. We have found out that certain simple transformation may have a negligible effect on phase shift but introduce a controllable inelasticity. This transformation does not change the real part of the potential but adds an imaginary part. As a result we get an optical potential with energy independent real part and energy dependent imaginary part. We apply this scheme to NN , π^-N and K^+N scattering successfully.

6 Appendix

We present only nontrivial integrals of (28). They can be derived from the recursion relations for the Riccati-Hankel functions and from known integrals [26].

$$I_1(x, \beta_i, \beta_k, l) = \int_x^\infty h_l^+(\beta_i t) h_l^+(\beta_k t) dt = \begin{cases} -x((h_l^+(\beta_i x))^2 - h_{l-1}^+(\beta_i x) h_{l+1}^+(\beta_i x))/2, & \text{for } i = k \\ \frac{\beta_i h_{l-1}^+(\beta_i x) h_l^+(\beta_k x) - \beta_k h_l^+(\beta_i x) h_{l-1}^+(\beta_k x)}{\beta_{ik}}, & \text{for } i \neq k. \end{cases} \quad (64)$$

$$\int_x^\infty h_l^+(\beta_i t) h_l^+(\beta_k t) t dt = \begin{cases} -\left(x(h_l^+(\beta_i x))^2 + I_1(x, \beta_i, \beta_i, l)\right)/(2\beta_i), & \text{for } i = k \\ \frac{2\beta_i}{\beta_{ik}^2}(\beta_k h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) - \beta_i h_{l-1}^+(\beta_i x) h_l^+(\beta_k x)) + \\ \quad + \frac{1}{\beta_{ik}} h_{l-1}^+(\beta_i x) h_l^+(\beta_k x) - \\ -\frac{x}{\beta_{ik}}(\beta_k h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) - \beta_i h_{l-1}^+(\beta_i x) h_l^+(\beta_k x)), & \text{for } i \neq k. \end{cases} \quad (65)$$

Where $\beta_{ik} = \beta_i^2 - \beta_k^2$

$$\int_x^\infty h_l^+(\beta_i t) h_l^+(\beta_k t) t^2 dt = \begin{cases} I_2(\beta_i x, l) / \beta_i^3, & \text{for } i = k \\ \frac{8\beta_i \beta_k}{\beta_{ik}^3}(\beta_k h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) - \beta_i h_l^+(\beta_k x) h_{l-1}^+(\beta_i x)) + \\ \quad + 2\frac{\beta_i}{\beta_{ik}^2}(h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) + \beta_k x h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) - \\ \quad - \beta_i x h_{l-1}^+(\beta_k x) h_l^+(\beta_i x)) + \\ \quad + \frac{x}{\beta_{ik}}(h_l^+(\beta_k x) h_{l-1}^+(\beta_i x) - h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) - \\ \quad - \beta_k x h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) + \beta_i x h_{l-1}^+(\beta_k x) h_l^+(\beta_i x)) - \\ \quad + \frac{2\beta_k}{\beta_{ik}^2}(h_l^+(\beta_k x) h_{l-1}^+(\beta_i x) - \beta_k x h_l^+(\beta_i x) h_{l-1}^+(\beta_k x) + \\ \quad + \beta_i x h_{l-1}^+(\beta_k x) h_l^+(\beta_i x)), & \text{for } i \neq k. \end{cases} \quad (66)$$

$$I_2(z, 1) = (iz^2/2 - 3z/2 - 9i/2 + 1/z) \exp(iz) \quad (67)$$

$$I_2(z, 2) = (-iz^2/2 + 7z/2 + 49i/2 - 24/z - 24i/x^2 + 12/x^3) \exp(iz) \quad (68)$$

$$I_2(z, 3) = (iz^2/2 - 13z/2 - 169i/2 + 171/z + 450i/x^2 - 765/x^3 - 810i/x^4 + 405/x^5) \exp(iz) \quad (69)$$

$$I_2(z, 4) = (-iz^2/2 + 21z/2 + 441i/2 - 745/z - 3510i/x^2 + 11835/x^3 + 28560i/x^4 - 47880/x^5 - 50400i/x^6 + 25200/x^7) \exp(iz) \quad (70)$$

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Table 1. The deuteron properties

	Exp. ^a	Calculation with Moscow potential
Energy (MeV)	2,22458900(22)	2,2246 ^a
Q (Fm ²)	0,2859(3)	0,277 ^c
A_S (Fm ^{-1/2})	0,8802(20)	0,8802
r_d (Fm)	1,9627(38)	1,956
$\eta_{d/s}$	0,02714	0,02714
μ_d	0,857406(1)	0,859 ^c

^a relativistic correction included; ^b Data are from [21]; ^c Meson exchange currents are not included.

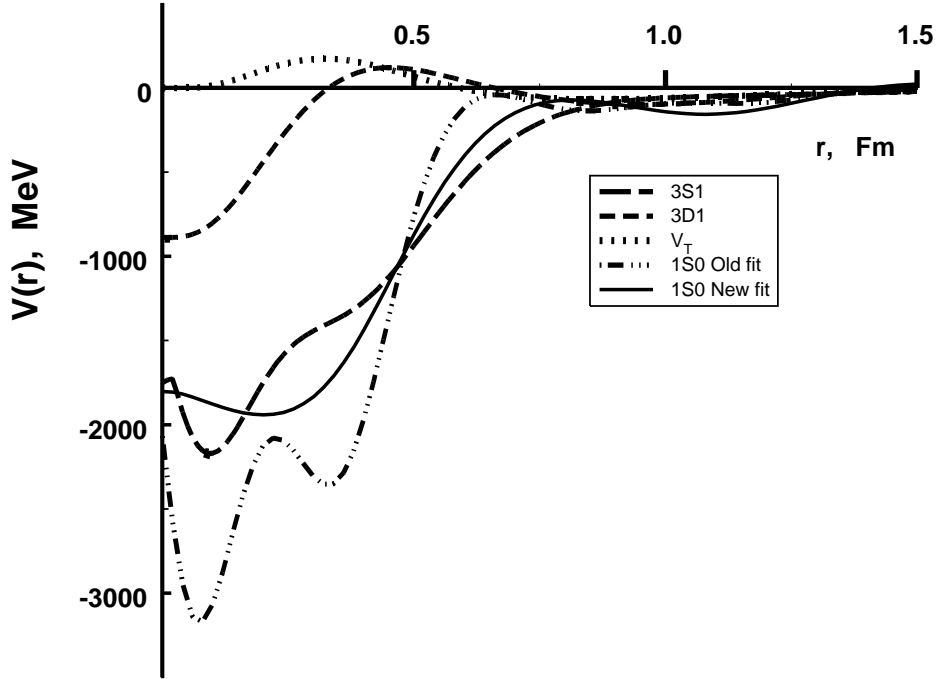


Figure 1: Real parts of NN potentials. Solid line 1S_0 -one channel (inversion from new fit of fig. 2). Dash-three points line 1S_0 -one channel (inversion from old fit of fig. 3). Two bound channels long dashed line $V_{CS}(r)$ (3S_1), short dashed line $V_{CD}(r)$ (3D_1), dotted line $V_{tens}(r)$.

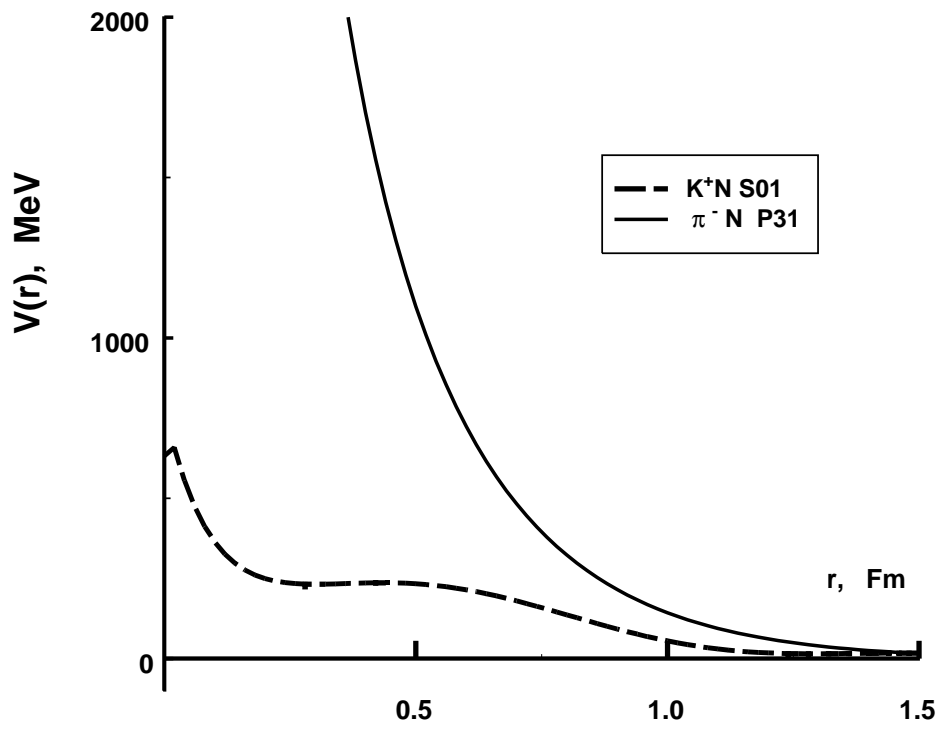


Figure 2: Real parts of π^-N P31 and K^+N S01 potentials.

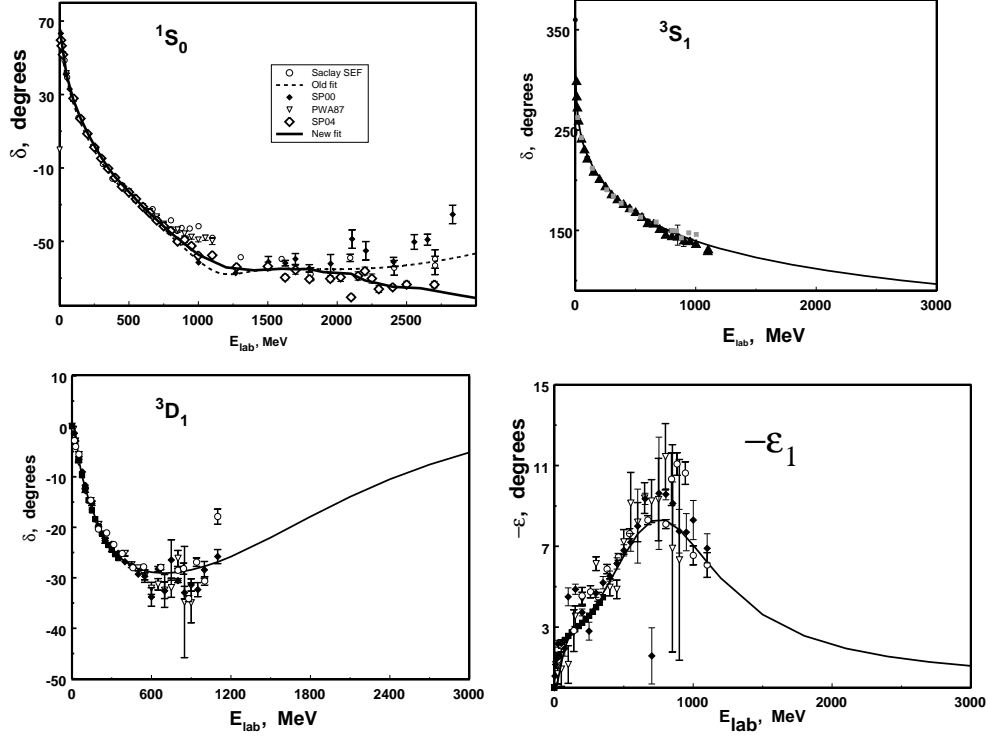


Figure 3: Phase shifts and mixing parameter. Results for optical potentials are indistinguishable from that as for real potentials. Solid lines, reconstructed Moscow potential; dashed lines reconstructed repulsive core potential. The phase shift analysis data are from [19, 20]. For S waves the original data set from [19, 20] is raised 180 degrees up. To leave the S -matrix unchanged we must then change the sign of the mixing parameter ϵ_1 for the Moscow potential. SP00 and SP04 data are from [20].

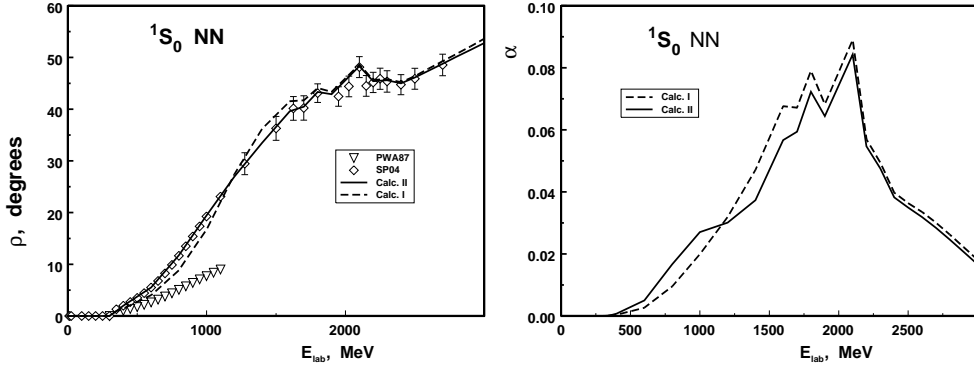


Figure 4: Left: The inelasticity parameters. The phase shift analysis data (circles) are from [19]. Right: parameter α .

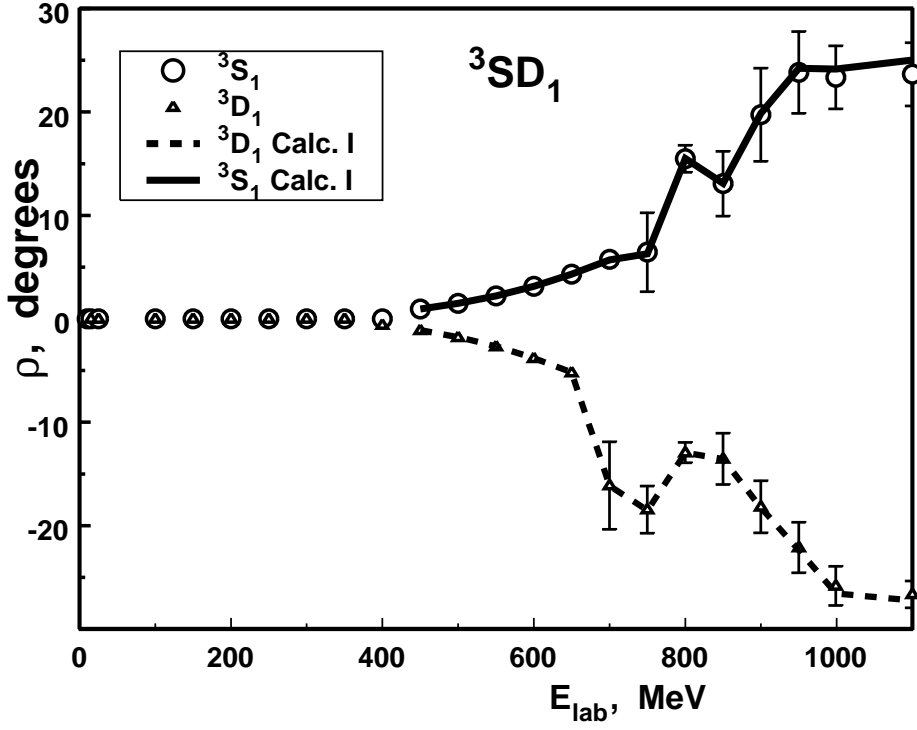


Figure 5: The inelasticity parameters ρ for 3SD_1 waves. The phase shift analysis data (circles) are from [20].

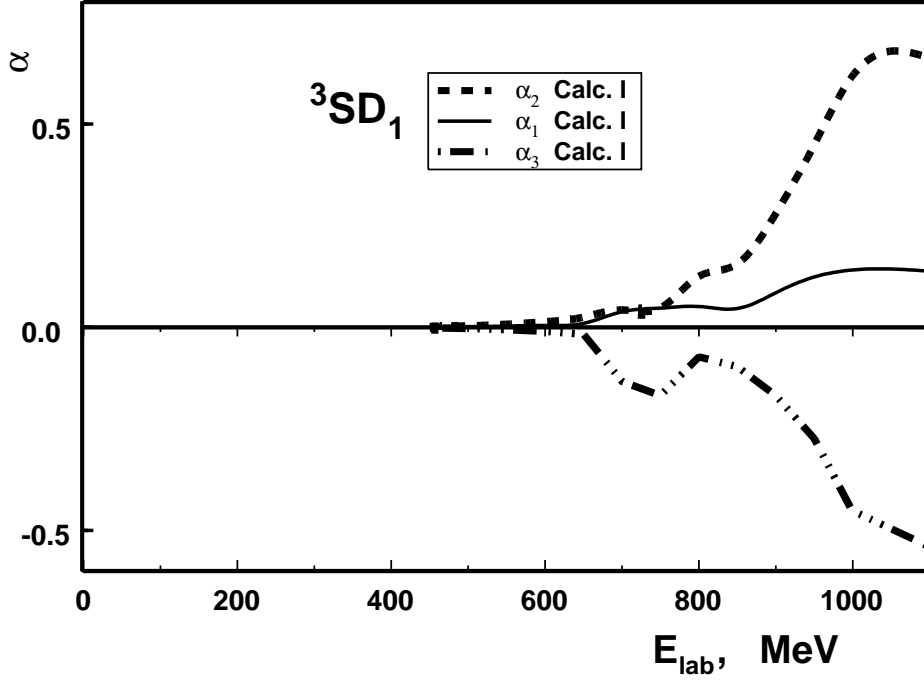


Figure 6: Parameters α_i for 3SD_1 NN potential predicted by eqs. (56) and (57). Solid line, α_1 ; dashed line with circles, α_2 ; dotted line, α_3 .

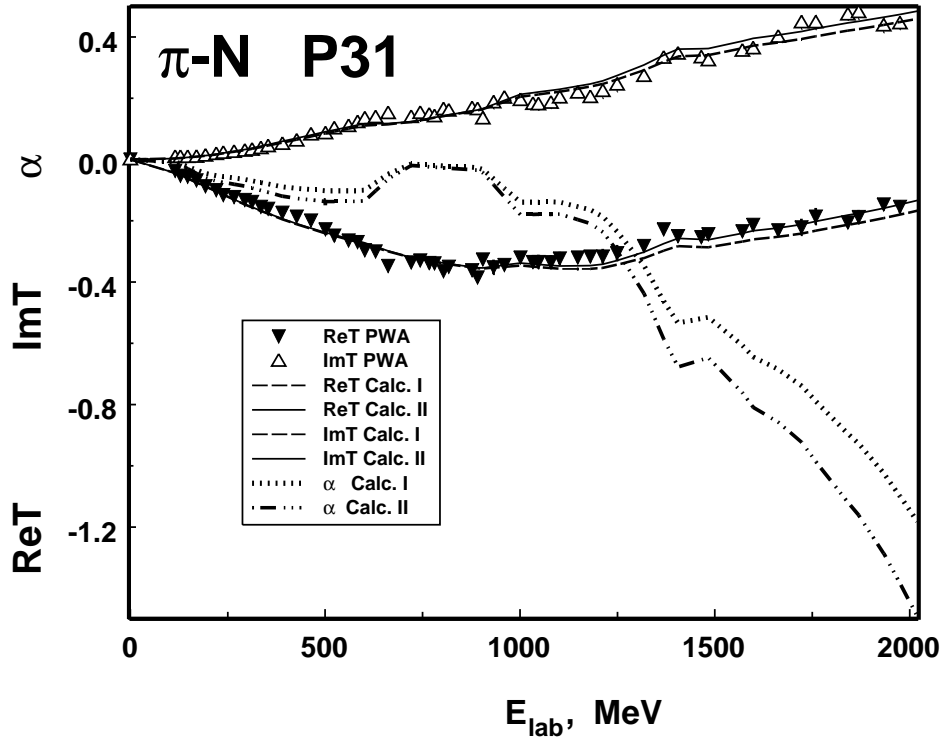


Figure 7: T -matrix and α for $P31 \pi^- N$, data are from [20].

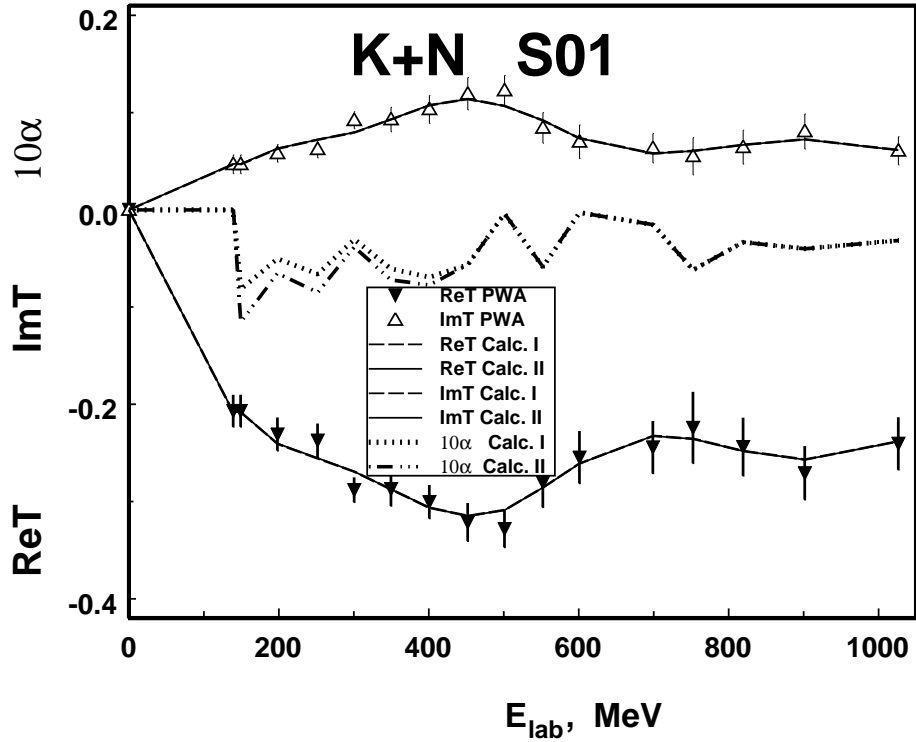


Figure 8: T -matrix and α for $S01 K^+ N$, data are from [20].